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On the Chern Character of a Theta-Summable Fredholm Module*

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In [3], Connes defines the notion of a theta-summable Fredholm module over a Banach algebra $\mathbf{A}$ with identity. This consists of a $\mathbf{Z}_2$-graded Hilbert space $\mathbf{H}=\mathbf{H}^+\oplus\mathbf{H}^-$ carrying a continuous representation of $\mathbf{A}$, and an odd self-adjoint operator $\mathscr{D}$: $\mathbf{H}^\pm\to\mathbf{H}^\mp$ with the following properties:

(1) if $a\in\mathbf{A}$, the operator $[\mathscr{D}, a]$ is densely defined and extends to a bounded operator on $\mathbf{H}$, and there is a constant $N(\mathscr{D})$ such that $\|a\|+\|[\mathscr{D},a]\|\leqslant N(\mathscr{D})\,\|a\|_{\mathbf{A}}$;

(2) for some $\varepsilon>0$, $\operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}$ is finite.

Here and in the rest of this paper, $\|A\|$ denotes the norm of A as a bounded operator on $\mathbf{H}$. One of the most important examples of a theta-summable Fredholm module is defined over the algebra $\mathbf{A}=C^1(M)$ of differentiable functions on a compact spin-manifold M: $\mathbf{H}$ is the space of L^2-sections of the spinor bundle $S^\pm$, with $\mathbf{A}$ acting by multiplication, and $\mathscr{D}$ is the Dirac operator.

Two different formulas have been proposed for the Chern character of a theta-summable Fredholm module, by Connes [3], and by Jaffe, Lesniewski, and Osterwalder [6]. In this paper, we show that if we adopt the second of these formulas as the definition of the Chern character, it is possible to simplify Connes's theory of the Chern character from K-homology of a Banach algebra to entire cyclic cohomology in a number of ways.

If $\mathbf{A}$ is an ungraded Banach algebra with identity, and $\bar{\mathbf{A}}$ is the Banach space $\mathbf{A}/\mathbf{C}$, we let $C_*(\mathbf{A})$ be the entire normalized bar complex of $\mathbf{A}$ defined by Connes [3]; this is the $\mathbf{Z}$-graded LF space obtained by taking the union of the completions of the space

$$\sum_n \mathbf{A}\otimes\bar{\mathbf{A}}^{\otimes n}$$

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with respect to the collection of seminorms

$$\left\| \sum_n A_n \right\|_z = \sum_{n=0}^{\infty} \frac{z^n \|A_n\|_\pi}{\sqrt{n!}}, \qquad z \in \mathbf{N}.$$

Here, $\|\cdot\|_\pi$ is the projective tensor product norm on $\mathbf{A} \otimes \bar{\mathbf{A}}^{\otimes n}$. The projective norm is characterized as follows: the continuous dual $(\mathbf{A} \otimes_\pi \bar{\mathbf{A}}^{\otimes_\pi n})'$ is isomorphic to the space of continuous bilinear maps $f\colon \mathbf{A} \times \bar{\mathbf{A}}^{\times n} \to \mathbf{C}$. The completion of $\mathbf{A} \otimes \bar{\mathbf{A}}^{\otimes n}$ in this topology will be denoted $C_n(\mathbf{A})$, and we will denote the element $a_0 \otimes \cdots \otimes a_n$ of $C_n(\mathbf{A})$ by $(a_0, ..., a_n)_n$, where $a_i \in \mathbf{A}$.

Note that our normalization conventions differ from those of Connes: our chain $\sum_n A_n$ corresponds in his normalization to the chain $\sum_n A_n/n!$.

Consider the two bounded operators on $C_*(\mathbf{A})$

$$b(a_0, ..., a_n)_n = \sum_{i=0}^{n-1} (-1)^i (a_0, ..., a_i a_{i+1}, ..., a_n)_{n-1} + (-1)^n (a_n, ..., a_{n-1})_{n-1}$$

and

$$B(a_0, ..., a_n)_n = \sum_{i=0}^{n} (-1)^{in} (1, a_i, ..., a_n, a_0, ..., a_{i-1})_{n+1}.$$

Note that $b^2 = B^2 = [b, B] = bB + Bb = 0$. Connes defines the entire cyclic homology $\mathrm{HE}_*(\mathbf{A})$ of $\mathbf{A}$ to be the homology of the complex $(C_*(\mathbf{A}), b + B)$. The operator $b + B$ is inhomogeneous, so that $\mathrm{HE}_*(\mathbf{A})$ is only $\mathbf{Z}_2$-graded; we denote the even subspace by HE_+ and the odd subspace by HE_-.

The cobar complex $C^*(\mathbf{A})$ is the topological dual $(C_*(\mathbf{A}))'$ of the bar complex; this is the same thing as the space of continuous multi-linear forms on $\mathbf{A} \times \bar{\mathbf{A}}^{\otimes n}$. This space carries two boundaries obtained by forming the adjoints of b and B acting on $C_*(\mathbf{A})$; we will denote these operators by the letters b and B too. The cohomology of the complex $(C^*(\mathbf{A}), b + B)$ is called the entire cyclic cohomology of $\mathbf{A}$, and is denoted $\mathrm{HE}^*(\mathbf{A})$. The pairing between $C^*(\mathbf{A})$ and $C_*(\mathbf{A})$ induces a pairing between $\mathrm{HE}^*(\mathbf{A})$ and $\mathrm{HE}_*(\mathbf{A})$, which we write $\langle \cdot, \cdot \rangle\colon \mathrm{HE}^*(\mathbf{A}) \otimes \mathrm{HE}_*(\mathbf{A}) \to \mathbf{C}$.

Let $(\mathbf{H}, \mathscr{D})$ be a theta-summable Fredholm module over $\mathbf{A}$. In (2.3), we give the definition, due to Jaffe, Lesniewski, and Osterwalder, of the Chern character $\mathrm{Ch}^*(\mathscr{D}) \in C^+(\mathbf{A})$ of this Fredholm module; part (1) of the following theorem is due to them.

THEOREM A. (1) *The Chern character of a theta-summable Fredholm module is closed*:

$$(b + B)\,\mathrm{Ch}^*(\mathscr{D}) = 0.$$

(2) *If* $(\mathbf{H}, \mathscr{D}_\tau)$ *is a differentiable one-parameter family of Fredholm modules over* $\mathbf{A}$ *such that* $\dot{\mathscr{D}}_\tau$ *is bounded, there is a cochain* $\widetilde{\mathrm{Ch}}^*(\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau) \in C^-(\mathbf{A})$ *such that*

$$(b+B)\,\widetilde{\mathrm{Ch}}^*(\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau) = \frac{d\,\mathrm{Ch}^*(\mathscr{D}_\tau)}{d\tau}.$$

(3) *If* $\mathrm{Tr}\, e^{-t\mathscr{D}^2} \leqslant \infty$ *for all* $t > 0$, *and if* $\mathscr{D}_\tau = \tau\mathscr{D}$ *for* $\tau > 0$,

$$(b+B)\,\widetilde{\mathrm{Ch}}^*(\mathscr{D}_\tau, \mathscr{D}) = \frac{d\,\mathrm{Ch}^*(\mathscr{D}_\tau)}{d\tau}.$$

Thus, in these two cases, the class of $\mathrm{Ch}^*(\mathscr{D}_\tau)$ *in* $\mathrm{HE}^+(\mathbf{A})$ *is independent of the parameter* τ.

An element of the space $K_0(\mathbf{A})$ may be represented by an idempotent $p \in M_r(\mathbf{A})$, that is, an operator satisfying $p^2 = p$. In Section 1, we will define a Chern character map from idempotents p to cyclic chains $\mathrm{Ch}_*(p) \in C_+(\mathbf{A})$, which has the advantage over Connes's choice that it is closed under the boundary $b + B$; this Chern character is given by the same formula as that of Hood and Jones [5, p. 362]. The analogue for $\mathrm{Ch}_*(p)$ of Theorem A is the following theorem, which we prove in Section 1.

THEOREM B. *Given an idempotent* p, *its Chern character* $\mathrm{Ch}_*(p) \in C_+(\mathbf{A})$ *satisfies*

$$(b+B)\,\mathrm{Ch}_*(p) = 0.$$

(2) *If* $p_\tau\colon [0,1] \to M_r(\mathbf{A})$ *is a one-parameter family of idempotents, there is a chain* $\widetilde{\mathrm{Ch}}_*(p_\tau, \dot{p}_\tau) \in C_-(\mathbf{A})$ *such that*

$$(b+B)\,\widetilde{\mathrm{Ch}}_*(p_\tau, \dot{p}_\tau) = \frac{d\,\mathrm{Ch}_*(p_\tau)}{d\tau}.$$

Thus, the homology class of the Chern character in $\mathrm{HE}_+(\mathbf{A})$ *is independent of* τ.

In the course of this paper, the following theorem, proved in Section 2, turns out to be quite useful; it shows that Fredholm modules are stable under bounded perturbations.

THEOREM C. *If* $(\mathbf{H}, \mathscr{D})$ *is a Fredholm module over* $\mathbf{A}$ *and* V *is an odd self-*

adjoint bounded operator on $\mathbf{H}$, *then* $(\mathbf{H}, \mathscr{D}+V)$ *is a theta-summable Fredholm module, and*

$$\operatorname{Tr} e^{-(1-\varepsilon/2)(\mathscr{D}+V)^2} \leqslant e^{(1+2/\varepsilon)\|V\|^2} \cdot \operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}.$$

Finally, in Section 3, we prove the following index formula.

THEOREM D. *If* $(\mathbf{H}, \mathscr{D})$ *is a Fredholm module over* $\mathbf{A}$ *and* $p \in M_r(\mathbf{A})$ *is an idempotent, let* $\mathscr{D}_p$ *be the Fredholm operator* $p \cdot \mathscr{D} \cdot p$ *on the Hilbert space* $p[\mathbf{H} \otimes \mathbf{C}^r]$. *If we denote the index of* $\mathscr{D}_p^+ : p[\mathbf{H}^+ \otimes \mathbf{C}^r] \to p[\mathbf{H}^- \otimes \mathbf{C}^r]$ *by* $\operatorname{ind}(\mathscr{D}_p)$, *then the following formula holds*:

$$(\mathrm{Ch}^*(\mathscr{D}), \mathrm{Ch}_*(p)) = \operatorname{ind}(\mathscr{D}_p).$$

It is possible to generalize the notion of a theta-summable Fredholm module by replacing the condition that $\mathscr{D}$ is self-adjoint by the condition that $\mathscr{D} - \mathscr{D}^*$ is bounded. All of the results of this paper may be generalized to this setting; however, the proofs become more cumbersome, since we can no longer use the spectral theorem to perform the necessary estimates, and must instead make use of the perturbation theory of analytic semigroups and Trotter's product formula.

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1. The Chern Character of an Idempotent

Consider an idempotent $p = (p_{ij}) \in M_r(\mathbf{A})$, which represents the class $[p] \in K_0(\mathbf{A})$. The following two chains are the only elements of $C_n(\mathbf{A})$ that can be defined using the matrices p and 1 and the standard trace on $M_r(\mathbf{A})$:

$$p_n = \operatorname{Tr}(p, ..., p)_n = \sum_{i_0 \cdots i_n} (p_{i_0 i_1}, p_{i_1 i_2}, ..., p_{i_k i_0})_n$$

$$q_n = \operatorname{Tr}(1, p, ..., p)_n = \sum_{i_1 \cdots i_n} (1, p_{i_1 i_2}, ..., p_{i_k i_1})_n.$$

Since $p^2 = p$, the boundaries of these chains are as follows if n is even,

$$bp_n = p_{n-1}$$

$$bq_n = 2p_{n-1} - q_{n-1}$$

$$Bp_n = (n+1) q_{n+1}$$

$$Bq_n = 0$$

while if n is odd, $bp_n = bq_n = Bp_n = Bq_n = 0$.

PROPOSITION 1.1. *Define the Chern character in* $C_*(\mathrm{A})$ *of* p *by the formula*

$$\mathrm{Ch}_*(p) = p_0 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \left(p_{2k} - \frac{1}{2} q_{2k} \right)$$
$$= (p)_0 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \left(p - \frac{1}{2}, p, \ldots, p \right)_{2k}.$$

Then $(b+B)\,\mathrm{Ch}_*(p) = 0$, *and linear combinations of the form* $s\,\mathrm{Ch}_*(p) + t\,\mathrm{Ch}_*(1)$, s, $t \in C$, *are the only combinations* of p_{2k} *and* q_{2k} *which have this property.* (*Note that* $\mathrm{Ch}_*(1)$ *equals* q_0.)

Proof. It is easy to see that $\mathrm{Ch}_*(p)$ satisfies the estimates for it to be a member of $C_*(\mathrm{A})$. To show that this chain and q_0 are the unique closed combinations in the elements p_{2k} and q_{2k}, let us write out the arbitrary linear combination of p_{2k} and q_{2k}:

$$\sum_{k=0}^{\infty} (s_k p_{2k} + t_k q_{2k}).$$

Applying the boundary $b+B$, we obtain

$$\sum_{k=1}^{\infty} ((s_k + 2t_k) p_{2k-1} + ((2k-1)s_{k-1} - t_k) q_{2k-1}) = 0.$$

It follows that $s_k = -2t_k$ for all $k > 0$, and that $s_k = -2(2k-1)s_{k-1}$, which has two independent solutions, in the first of which $t_0 = 0$ and $s_k = (-1)^k (2k)! s_0/k!$, and in the second of which t_0 is arbitrary and $s_k = 0$ for all k. ∎

Note that the coefficients of the chains p_{2k} and q_{2k} in $\mathrm{Ch}_*(p)$ are all integers.

Connes [2] has introduced the following chain, which he calls the Chern character of the idempotent p:

$$\mathrm{Ch}_*(p) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} p_{2k}.$$

This character has the disadvantage that it is not closed with respect to the boundary operator $b+B$, which leads Connes to a certain amount of difficulty in developing his theory.

We shall now investigate the behaviour of the Chern character $\mathrm{Ch}_*(p)$

under a differentiable homotopy of the idempotent p. If p_τ: $[0, 1] \to M_r(\mathbf{A})$ is a differentiable one-parameter family of idempotents, then

$$\frac{dp_\tau}{d\tau} = [a_\tau, p_\tau] \qquad \text{where} \quad a_\tau = \dot{p}_\tau(2p_\tau - 1).$$

Indeed, this follows from the formula $p_\tau \dot{p}_\tau + \dot{p}_\tau p_\tau = \dot{p}_\tau$.

We now make use of the following lemma. If $a \in \mathbf{A}$, let $\iota(a)$: $C_n(\mathbf{A}) \to C_{n+1}(\mathbf{A})$ be the map of the bar complex defined by

$$\iota(a) \cdot (a_0, ..., a_n)_n = \sum_{i=0}^{n} (-1)^i (a_0, ..., a_i, a, a_{i+1}, ..., a_n)_{n+1}.$$

Also, let $L(a)$: $C_n(\mathbf{A}) \to C_n(\mathbf{A})$ denote the operator

$$L(a) \cdot (a_0, ..., a_n)_n = \sum_{i=0}^{n} (a_0, ..., [a, a_i], ..., a_n)_n.$$

Note that $\iota(a)$ is an odd operator; thus, by $[b, \iota(a)]$ and $[B, \iota(a)]$, we mean the anticommutator. The following lemma shows that the operator $L(a)$ acts by zero on Hochschild and cyclic homology; its proof is a straightforward combinatoric exercise.

LEMMA 1.2. *The operators $\iota(a)$ and $L(a)$ are bounded on $C_*(\mathbf{A})$, and satisfy the following formulas*:

(1) $[b, \iota(a)] = L(a)$

(2) $[B, \iota(a)] = 0$.

Since $[b + B, \iota(a)] = L(a)$, we obtain the following result, completing the proof of Theorem B.

PROPOSITION 1.3. *Let $\widetilde{\mathrm{Ch}}_*(p, q)$ denote the odd cyclic chain $\widetilde{\mathrm{Ch}}_*(p, q) = \iota(q(2p-1)) \cdot \mathrm{Ch}_*(p)$. Then*

$$\frac{d\,\mathrm{Ch}_*(p_\tau)}{d\tau} = (b + B)\, \widetilde{\mathrm{Ch}}_*(p_\tau, \dot{p}_\tau).$$

Proof. Since $L(a_\tau) \cdot p_n = \dot{p}_n$ and $L(a_\tau) \cdot q_n = \dot{q}_n$, it follows that

$$\frac{d\,\mathrm{Ch}_*(p_\tau)}{d\tau} = L(a_\tau) \cdot \mathrm{Ch}_*(p_\tau).$$

Lemma 1.2 and the fact that $(b + B)\,\mathrm{Ch}_*(p_\tau) = 0$ combine to complete the proof. ∎

2. The Chern Character of a Fredholm Module

In this section, we will study the definition of the Chern character of a theta-summable Fredholm module given in [6]. In order to define the Chern character, it is helpful to introduce the following notation. Let Δ_n be the n-simplex

$$\{(t_1, ..., t_n) \in \mathbf{R}^n \mid 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1\}.$$

If A_i, $0 \leqslant i \leqslant n$, are operators on $\mathbf{H}$, we define

$$\langle A_0, ..., A_n \rangle_n = \int_{\Delta_n} \operatorname{Str}(A_0 e^{-t_1 \mathscr{D}^2} A_1 e^{-(t_2 - t_1)\mathscr{D}^2} \cdots A_n e^{-(1-t_n)\mathscr{D}^2}).$$

Here the supertrace $\operatorname{Str} A$ of an operator A on $\mathbf{H}$ is defined to be $\operatorname{Str} A = \operatorname{Tr} A|_{\mathbf{H}^+} - \operatorname{Tr} A|_{\mathbf{H}^-}$. The following lemma gives an estimate for $\langle A_0, ..., A_n \rangle_n$.

Lemma 2.1. *If the operators A_i and B_i are bounded, and at most k of the operators A_i are non-zero, then*

$$|\langle A_0 \mathscr{D} + B_0, ..., A_n \mathscr{D} + B_n \rangle_n| \leqslant \frac{\varepsilon^{-k/2} \operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}}{(n-k)!} \prod_{i=0}^{n} (\|A_i\| + \|B_i\|),$$

where $\varepsilon > 0$ is a constant such that $\operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2} < \infty$.

Proof. These bounds are a simple consequence of the Hölder inequality for the trace on a Hilbert space: if $\|A\|_p$ denotes the p-Schauder norm of the operator A, and $\sigma_0 + \cdots + \sigma_n = 1$, we have

$$|\operatorname{Tr}(A_0 \cdots A_n)| \leqslant \|A_0\|_{\sigma_0^{-1}} \cdots \|A_n\|_{\sigma_n^{-1}}.$$

If we change variables on the simplex from $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1$ to $\sigma_0 + \cdots + \sigma_n = 1$, where $\sigma_i = t_{i+1} - t_i \geqslant 0$, we see that

$$\begin{aligned} &|\langle A_0 \mathscr{D} + B_0, ..., A_n \mathscr{D} + B_0 \rangle_n| \\ &\quad \leqslant \int_{\Delta_n} |\operatorname{Tr}((A_0 \mathscr{D} + B_0) e^{-\sigma_0 \mathscr{D}^2} \cdot \cdot (A_n \mathscr{D} + B_n) e^{-\sigma_n \mathscr{D}^2})| \\ &\quad \leqslant \int_{\Delta_n} \|(A_0 \mathscr{D} + B_0) e^{-\sigma_0 \mathscr{D}^2}\|_{\sigma_0^{-1}} \cdots \|(A_n \mathscr{D} + B_n) e^{-\sigma_n \mathscr{D}^2}\|_{\sigma_n^{-1}}. \end{aligned}$$

In estimating $\|(A\mathscr{D} + B) e^{-\sigma \mathscr{D}^2}\|_{\sigma^{-1}}$, we use the facts that

$$\begin{aligned} \|A \mathscr{D} e^{-\sigma \mathscr{D}^2}\|_{\sigma^{-1}} &\leqslant \|A\| \cdot \|\mathscr{D} e^{-\varepsilon \sigma \mathscr{D}^2}\| \cdot \|e^{-(1-\varepsilon)\sigma \mathscr{D}^2}\|_{\sigma^{-1}} \\ &\leqslant (2e\varepsilon\sigma)^{-1/2} (\operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2})^{\sigma} \cdot \|A\|, \end{aligned}$$

and that

$$\|Be^{-\sigma\mathscr{D}^2}\|_{\sigma^{-1}} \leqslant \|B\| \cdot \|e^{-\sigma\mathscr{D}^2}\|_{\sigma^{-1}} \leqslant (\operatorname{Tr} e^{-\mathscr{D}^2})^{\sigma} \cdot \|B\|.$$

The proof is finished by using the bound $\int_{\Delta_n} (\sigma_0 \cdots \sigma_{k-1})^{-1/2} \leqslant 2^k/(n-k)!$. ∎

Using the fact that

$$\langle A_0, ..., A_i\mathscr{D}, A_{i+1}, ..., A_n\rangle = \langle A_0, ..., A_i, \mathscr{D}A_{i+1}, ..., A_n\rangle,$$

we obtain a number of bounds that are not explicitly contained in the above lemma. For example, if the operators A_i are all bounded, then we obtain the following bounds:

$$\langle A_0, ..., [\mathscr{D}, A_i], ..., A_n\rangle_n \leqslant \frac{2\varepsilon^{-1/2} \operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}}{(n-1)!} \|A_0\| \cdots \|A_n\| \quad (2.1a)$$

$$\langle A_0, ..., \mathscr{D}^2, ..., A_n\rangle_{n+1} \leqslant \frac{\varepsilon^{-1} \operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}}{(n-1)!} \|A_0\| \cdots \|A_n\|. \quad (2.1b)$$

The following lemma collects some useful formulas involving the forms $\langle A_0, ..., A_n\rangle_n$; parts of it are borrowed from [4, 6]. In the statement of the lemma, we write $|A| \in \mathbf{Z}_2$ for the degree of an operator A acting on the graded Hilbert space $\mathbf{H}$; that is, $|A| = 0$ if A is even, and $|A| = 1$ if A is odd.

LEMMA 2.2. *In each of the following cases, we assume that the operators A_i are such that each term is well-defined.*

(1) $\langle A_0, ..., A_n\rangle_n = (-1)^{(|A_0| + \cdots + |A_{i-1}|)(|A_i| + \cdots + |A_n|)} \times \langle A_i, ..., A_n, A_0, ..., A_{i-1}\rangle_n$

(2) $\langle A_0, ..., A_n\rangle_n = \sum_{i=0}^{n} (-1)^{(|A_0| + \cdots + |A_{i-1}|)(|A_i| + \cdots + |A_n|)} \times \langle 1, A_i, ..., A_n, A_0, ..., A_{i-1}\rangle_{n+1}$

(3) $\sum_{i=0}^{n} (-1)^{|A_0| + \cdots + |A_{i-1}|} \langle A_0, ..., [\mathscr{D}, A_i], ..., A_n\rangle_n = 0$

(4) $\langle A_0, ..., [\mathscr{D}^2, A_i], ..., A_n\rangle_n = \langle A_0, ..., A_{i-1}A_i, A_{i+1}, ..., A_n\rangle_{n-1} - \langle A_0, ..., A_{i-1}, A_iA_{i+1}, ..., A_n\rangle_{n-1}$

(5) *If $\mathscr{D}_\tau$ is a differentiable one-parameter family of self-adjoint odd operators on* $\mathbf{H}$ *such that* $\operatorname{Tr} e^{-\mathscr{D}_\tau^2}$ *is uniformly bounded as t varies, we have*

$$\frac{d}{d\tau} \langle A_0, ..., A_n\rangle_n + \sum_{i=0}^{n} \langle A_0, ..., A_i, [\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau], A_{i+1}, ..., A_n\rangle_{n+1} = 0.$$

Proof. The cyclic symmetry in (1) follows from the fact that $\operatorname{Str}[A, B]=0$. To prove (2), we start from the equation

$$\langle A_0, \ldots, A_n\rangle_n=\int_{[0,1]\times\Delta_n} \operatorname{Str}(A_\Omega e^{-t_1\mathscr{D}^2}\cdots A_n e^{-(1-t_n)\mathscr{D}^2})\, ds\, dt_1\cdots dt_n.$$

We divide the region of integration into $n+1$ pieces

$$R_i=\{t_i\leqslant s\leqslant t_{i+1}\}.$$

Each of these regions R_i is a simplex, which contributes the term

$$\langle A_0, \ldots, A_i, 1, A_{i+1}, \ldots, A_n\rangle_{n+1}$$

to the sum, from which (2) follows.

Part (3) follows directly from the fact that

$$\operatorname{Str}[\mathscr{D}, A_0 e^{-t_1\mathscr{D}^2} A_1 e^{-(t_2-t_1)\mathscr{D}^2}\cdots A_n e^{-(1-t_n)\mathscr{D}^2}]=0.$$

To prove part (4) we observe that

$$[e^{-\mathscr{D}^2}, A]+\int_0^1 e^{-s\mathscr{D}^2}[\mathscr{D}^2, A]e^{-(1-s)\mathscr{D}^2}\, ds=0.$$

Replacing $\mathscr{D}^2$ by $(t_{i+1}-t_i)\mathscr{D}^2$, and using the substitution $u=(t_{i+1}-t_i)s+t_i$, we obtain

$$[e^{-(t_{i+1}-t_i)\mathscr{D}^2}, A_i]+\int_{t_i}^{t_{i+1}} e^{-(t_{i+1}-u)\mathscr{D}^2}[\mathscr{D}^2, A_i]e^{-(u-t_i)\mathscr{D}^2}\, du=0.$$

Inserting this into the definition of $\langle A_0, \ldots, [\mathscr{D}^2, A_i], \ldots, A_n\rangle_n$ gives the desired formula.

To prove part (5), observe that by Leibniz's rule,

$$\frac{d}{d\tau}\langle A_0, \ldots, A_n\rangle + \sum_{i=0}^{n} \operatorname{Str}\left(A_0 e^{-t_1\mathscr{D}^2}\cdots A_i \frac{d(e^{-(t_{i+1}-t_i)\mathscr{D}_\tau^2})}{d\tau} A_{i+1}\cdots A_n e^{-(1-t_n)\mathscr{D}^2}\right)=0. \quad (*)$$

We now recall Duhamel's equation

$$\frac{d(e^{-\mathscr{D}_\tau^2})}{d\tau}+\int_0^1 e^{-s\mathscr{D}_\tau^2}[\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau]e^{-(1-s)\mathscr{D}_\tau^2}\, ds=0.$$

Replacing $\mathscr{D}_\tau^2$ by $(t_{i-1}-t_i)\mathscr{D}_\tau^2$, and using the substitution $u=(t_{i+1}-t_i)s+t_i$, we obtain

$$\frac{d(e^{-(t_{i+1}-t_i)\mathscr{D}_\tau^2})}{d\tau}+\int_{t_i}^{t_{i+1}} e^{-(t_{i+1}-u)\mathscr{D}_\tau^2}[\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau]\, e^{-(u-t_i)\mathscr{D}_\tau^2}\, du=0.$$

The formula follows from substituting this into (∗). ∎

As a last preliminary result, we prove the stability of Fredholm modules under perturbation by a bounded operator. First we recall the fact that if A and B are positive self-adjoint operators, then

$$\operatorname{Tr} e^{-A-B} \leqslant \operatorname{Tr} e^{-A}. \tag{2.2}$$

Proof of Theorem C. Let $(\mathbf{H}, \mathscr{D})$ be a Fredholm module over $\mathbf{A}$, and let V be an odd self-adjoint bounded operator on $\mathbf{H}$. If $a \in \mathbf{A}$, then

$$\|[\mathscr{D}+V, a]\| \leqslant (N(\mathscr{D})+2\|V\|)\,\|a\|_{\mathbf{A}},$$

which is the first part of the definition of a theta-summable Fredholm module.

To prove that $\mathscr{D}+V$ has trace class heat kernel, introduce the operators

$$A=(1-\varepsilon)\mathscr{D}^2$$
$$B=\varepsilon\mathscr{D}^2/2+(1-\varepsilon/2)(\mathscr{D}V+V\mathscr{D}+V^2)+(1+2/\varepsilon)\|V\|^2.$$

Note that $A+B=(1-\varepsilon/2)(\mathscr{D}+V)^2+(1+2/\varepsilon)\|V\|^2$, and that A is a positive operator. To see that B is positive, we use the fact that

$$-(\mathscr{D}V+V\mathscr{D}) \leqslant \varepsilon\mathscr{D}^2/2+2V^2/\varepsilon \leqslant \varepsilon\mathscr{D}^2/2+2\|V\|^2/\varepsilon.$$

Thus, applying (2.2), we obtain

$$\operatorname{Tr} e^{-(1-\varepsilon/2)(\mathscr{D}+V)^2-(1+2/\varepsilon)\|V\|^2} \leqslant \operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}. \quad ∎$$

The Chern character of a theta-summable Fredholm module $(\mathbf{H}, \mathscr{D})$ of Jaffe, Lesniewski, and Osterwalder [6] is the even cochain $\mathrm{Ch}^*(\mathscr{D})$ on $\mathbf{A}$ defined by the formula

$$(\mathrm{Ch}^{2k}(\mathscr{D}), (a_0, \ldots, a_{2k})_{2k})=\langle a_0, [\mathscr{D}, a_1], \ldots, [\mathscr{D}, a_{2k}]\rangle_{2k}. \tag{2.3}$$

Note that the right-hand side vanishes if $a_i=1$ for $1 \leqslant i \leqslant 2k$, as it must in order to define a normalized cochain. The following result of [6] shows that $\mathrm{Ch}^*(\mathscr{D})$ defines an element of $\mathrm{HE}^+(\mathbf{A})$.

PROPOSITION 2.3. *The cochain* $\mathrm{Ch}^*(\mathscr{D})$ *is a closed element of* $C^*(\mathbf{A})$, *that is,* $(b+B)\,\mathrm{Ch}^*(\mathscr{D})$.

Proof. The proof that $\mathrm{Ch}^*(\mathscr{D})$ is an element of $C^*(\mathbf{A})$, is an immediate consequence of Lemma 2.1, which shows that

$$|(\mathrm{Ch}^{2k}, (a_0, ..., a_{2k})_{2k})| \leqslant \frac{N(\mathscr{D})^{2k+1} \operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}}{(2k)!} \|V\| \cdot \|a_0\|_{\mathbf{A}} \cdots \|a_{2k}\|_{\mathbf{A}}.$$

To prove that $(b+B)\,\mathrm{Ch}^*(\mathscr{D})=0$, we apply Lemma 2.2(3) with $A_0=a_0$ and $A_i=[\mathscr{D}, a_i]$ for $1 \leqslant i \leqslant 2k-1$. This leads to the following formula:

$$\begin{aligned}&\langle [\mathscr{D}, a_0], ..., [\mathscr{D}, a_{2k-1}] \rangle_{2k-1} \\ &\quad + \sum_{i=1}^{2k-1} (-1)^{i-1} \langle a_0, [\mathscr{D}, a_1], ..., [\mathscr{D}^2, a_i], ..., [\mathscr{D}, a_{2k-1}] \rangle_{2k-1} = 0.\end{aligned}$$

By Lemma 2.2(2), the first of these terms is just $(B\,\mathrm{Ch}^*(\mathscr{D}), (a_0, ..., a_{2k-1})_{2k-1})$. Using Lemma 2.2(4), we will show that the rest of this sum is equal to $(b\,\mathrm{Ch}^*(\mathscr{D}), (a_0, ..., a_{2k-1})_{2k-1})$. Indeed, the term

$$(-1)^{i-1} \langle a_0, [\mathscr{D}, a_1], ..., [\mathscr{D}^2, a_i], ..., [\mathscr{D}, a_{2k-1}] \rangle_{2k-1}$$

is equal to

$$\begin{aligned}&(-1)^{i-1} \langle a_0, [\mathscr{D}, a_1], ..., [\mathscr{D}, a_{i-1}] a_i, [\mathscr{D}, a_{i+1}], ..., [\mathscr{D}, a_{2k}] \rangle_{2k-1} \\ &\quad + (-1)^i \langle a_0, [\mathscr{D}, a_1], ..., [\mathscr{D}, a_{i-1}], a_i [\mathscr{D}, a_{i+1}], ..., [\mathscr{D}, a_{2k}] \rangle_{2k-1}.\end{aligned}$$

Adding all of this up and using the fact that $[\mathscr{D}, a_i a_{i+1}] = [\mathscr{D}, a_i] a_{i+1} + a_i [\mathscr{D}, a_{i+1}]$, we easily see that these terms do indeed conspire to give $b\,\mathrm{Ch}^*(\mathscr{D})$ evaluated on the chain $(a_0, ..., a_{2k-1})_{2k-1}$. ∎

We will now prove that the cohomology class defined by $\mathrm{Ch}^*(\mathscr{D})$ is invariant under a differentiable homotopy of the operator $\mathscr{D}$. The formulation of this is reminiscent of Proposition 1.3. To start with, if V is an operator on $\mathbf{H}$, let $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, V)$ be the cochain on $\mathbf{A}$ defined by the formula

$$\begin{aligned}&(\widetilde{\mathrm{Ch}}^*(\mathscr{D}, A), (a_0, ..., a_n)_n) \\ &\quad = \sum_{i=0}^{n} (-1)^{i|V|} \langle a_0, [\mathscr{D}, a_1], ..., [\mathscr{D}, a_i], V, [\mathscr{D}, a_{i+1}], ..., [\mathscr{D}, a_n] \rangle_{n+1}.\end{aligned} \tag{2.4}$$

Note that, by Lemma 2.2(1), $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, 1) = \mathrm{Ch}^*(\mathscr{D})$. Thus, the following result is actually a generalization of Proposition 2.3.

PROPOSITION 2.4. (1) *The cochain* $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, V)$ *is an element of* $C^*(\mathbf{A})$ *if either* (a) V *is bounded on* $\mathbf{H}$ or (b) $V=\mathscr{D}$.

(2) *The cochain* $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, V)\in C^+(\mathbf{A})$ *if* V *is even, and* $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, V)\in C^-(\mathbf{A})$ *if* V *is odd.*

(3) *Let* $\alpha^*(\mathscr{D}, V)$ *be the element of* $C^*(\mathbf{A})$ *defined by the formula*

$$(\alpha^*(\mathscr{D}, V), (a_0, ..., a_n)_n)$$
$$= \sum_{i=0}^{2k} (-1)^{(i-1)(|V|+1)}\langle a_0, [\mathscr{D}, a_1], ..., [V, a_i], ..., [\mathscr{D}, a_n]\rangle_n.$$

Then

$$(b+B)\,\widetilde{\mathrm{Ch}}^*(\mathscr{D}, V)+(-1)^{|V|}\,\widetilde{\mathrm{Ch}}^*(\mathscr{D}, [\mathscr{D}, V])+(-1)^{|V|}\alpha^*(\mathscr{D}, V)=0.$$

Proof. If V is a bounded operator on $\mathbf{H}$, we see from Lemma 2.1 that

$$|(\widetilde{\mathrm{Ch}}^n(\mathscr{D}, V), (a_0, ..., a_n)_n)| \leqslant \frac{(n+1)\,N(\mathscr{D})^{n+1}\operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}}{(n+1)!}\,\|a_0\|_{\mathbf{A}}\cdots\|a_n\|_{\mathbf{A}},$$

so that $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, V)$ is indeed in $C^*(\mathbf{A})$. On the other hand,

$$|(\widetilde{\mathrm{Ch}}^n(\mathscr{D}, \mathscr{D}), (a_0, ..., a_n)_n)|$$
$$\leqslant \frac{(n+1)\,N(\mathscr{D})^{n+1}\varepsilon^{-1/2}\operatorname{Tr} e^{-(1-\varepsilon)\mathscr{D}^2}}{n!}\,\|a_0\|_{\mathbf{A}}\cdots\|a_n\|_{\mathbf{A}},$$

and once more, we see that $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, \mathscr{D})\in C^*(\mathbf{A})$. To see that the parity of $\widetilde{\mathrm{Ch}}^*(\mathscr{D}, V)$ is the same as that of V, we use the fact that $\langle A, ..., A_n\rangle_n$ vanishes unless $|A_0|+\cdots+|A_n|=0$. The definition of $\widetilde{\mathrm{Ch}}^n(\mathscr{D}, V)$ shows us that in our case, this parity equals $|a_0|+\cdots+|a_n|+n+|V|=n+|V|$, proving (2).

The proof of (3) is similar to the proof that $(b+B)\,\mathrm{Ch}^*(\mathscr{D})=0$: we start by applying Lemma 2.2(3) with

$$A_j=\begin{cases} a_0, & j=0,\\ [\mathscr{D}, a_j], & j\leqslant i,\\ V, & j=i,\\ [\mathscr{D}, a_{j-1}], & j>i.\end{cases}$$

This gives

$$X_1+X_2+X_3=0, \tag{$**$}$$

where

$$X_1=(-1)^{i|V|}\langle[\mathscr{D},a_0],\ldots,[\mathscr{D},a_i],V,[\mathscr{D},a_{i+1}],\ldots,[\mathscr{D},a_n]\rangle_{n+1},$$

$$\begin{aligned}X_2=&\sum_{j<i}(-1)^{i|V|+j-1}\\&\times\langle a_0,[\mathscr{D},a_1],\ldots,[\mathscr{D}^2,a_j],\ldots,[\mathscr{D},a_i],V,\ldots,[\mathscr{D},a_n]\rangle_{n+1}\\&+\sum_{j>i}(-1)^{(i+1)|V|+j-1}\\&\times\langle a_0,[\mathscr{D},a_1],\ldots,[\mathscr{D},a_i],V,\ldots,[\mathscr{D}^2,a_j],\ldots,[\mathscr{D},a_n]\rangle_{n+1},\end{aligned}$$

$$\begin{aligned}X_3=&(-1)^{i(|V|+1)}\\&\times\langle a_0,[\mathscr{D},a_1],\ldots,[\mathscr{D},a_i],[\mathscr{D},V],[\mathscr{D},a_{i+1}],\ldots,[\mathscr{D},a_n]\rangle_{n+1}.\end{aligned}$$

Note that all of the terms in this formula are well-defined if either V is bounded or equals $\mathscr{D}$; this is seen by applying Lemma 2.1 or its corollaries (2.1a) and (2.1b).

We now sum (**) over $0\leqslant i\leqslant n$. By Lemma 2.2(2), we see after reordering terms that

$$\sum_i X_1=(-1)^{|V|}(B\widetilde{\mathrm{Ch}}^*(\mathscr{D},V),(a_0,\ldots,a_n)_n).$$

Similarly, using Lemma 2.2(4),

$$\sum_i X_2=((-1)^{|V|}b\,\widetilde{\mathrm{Ch}}^*(\mathscr{D},V)+\alpha^*(\mathscr{D},V),(a_0,\ldots,a_n)_n).$$

Finally, it is clear that

$$\sum_i X_3=(\widetilde{\mathrm{Ch}}^*(\mathscr{D},[\mathscr{D},V]),(a_0,\ldots,a_n)_n).\quad\blacksquare$$

We can now prove the main results of this section, which form parts (2) and (3) of Theorem A of the Introduction.

COROLLARY 2.5. *Let $\mathscr{D}_\tau$ be a one-parameter family of operators on* **H** *such that either*

(1) *the operators $\dot{\mathscr{D}}_\tau$ form a continuous family of bounded operators,*

or

(2) $\operatorname{Tr} e^{-t\mathscr{D}^2}$ *is finite for all $t>0$ and $\mathscr{D}_\tau=\tau\mathscr{D}$.*

Then we have the homotopy formula

$$\frac{d\,\mathrm{Ch}^*(\mathscr{D}_\tau)}{d\tau} = (b+B)\,\widetilde{\mathrm{Ch}}^*(\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau).$$

Proof. Observe that under both the above sets of assumptions, the heat kernel $e^{-\mathscr{D}_\tau^2}$ is uniformly trace-class for τ in a bounded interval: if $\dot{\mathscr{D}}_\tau$ is bounded, this follows from Theorem D, while if $\mathscr{D}_\tau = \tau\mathscr{D}$, this is part of the hypothesis. Thus, we may apply Lemma 2.2(5), which shows that

$$\frac{d\,\mathrm{Ch}^*(\mathscr{D}_\tau)}{d\tau} = \widetilde{\mathrm{Ch}}^*(\mathscr{D}_\tau, [\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau]) + \alpha^*(\mathscr{D}_\tau, \dot{\mathscr{D}}_\tau).$$

The formula is now an immediate consequence of Proposition 2.4.

3. The Index Formula

In this section, we prove Theorem D, following Connes's treatment [3] closely. That is, we will show that

$$(\mathrm{Ch}^*(\mathscr{D}), \mathrm{Ch}_*(p)) = \mathrm{ind}(\mathscr{D}_p). \tag{3.1}$$

We start with the case in which p acts as a self-adjoint operator on $\mathbf{H}$ and commutes with $\mathscr{D}$. Using $[\mathscr{D}, p] = 0$, it is easy to see that

$$\begin{aligned}(\mathrm{Ch}^*(\mathscr{D}), \mathrm{Ch}_*(p)) &= \langle p \rangle_0 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \left\langle p - \frac{1}{2}, [\mathscr{D}, p], ..., [\mathscr{D}, p] \right\rangle_{2k} \\ &= \mathrm{Str}(p \cdot e^{-\mathscr{D}^2}) = \mathrm{Str}(p \cdot e^{-\mathscr{D}_p^2} \cdot p) = \mathrm{Str}(e^{-\mathscr{D}_p^2})|_{p[\mathbf{H} \otimes \mathbf{C}^r]}.\end{aligned}$$

We can now apply the following lemma (known as the McKean–Singer formula) with $H = p[\mathbf{H} \otimes \mathbf{C}^r]$ and $D = p \cdot \mathscr{D} \cdot p$.

Lemma 3.1. *If D is a self-adjoint odd operator on a Hilbert space H with trace-class heat operator e^{-D^2}, then* $\mathrm{Str}(e^{-D^2}) = \mathrm{ind}(D)$, *where* $\mathrm{ind}(D)$ *is the index of the operator* $D^+: H^+ \to H^-$.

This completes the proof of (3.1) in this special case.

We now observe that the two sides of (3.1) are homotopy invariant as functions of $\mathscr{D}$ and p. For $\mathrm{ind}(\mathscr{D}_p)$, this follows from Fredholm theory, while for $(\mathrm{Ch}^*(\mathscr{D}), \mathrm{Ch}_*(p))$, it is a consequence of Theorems A and B. Thus, in proving the theorem, we are free to choose $\mathscr{D}$ and p arbitrarily within their homotopy classes. We will use this freedom to arrange for p to be self-adjoint and to commute with $\mathscr{D}$.

Introduce the algebra $\mathbf{A}(\mathscr{D})$, which consists of all bounded operators a on $\mathbf{H}$ such that $[\mathscr{D}, a]$ is bounded. This is a Banach $*$-algebra with the norm $\|a\| + \|[\mathscr{D}, a]\|$, and $(\mathbf{H}, \mathscr{D})$ is a theta-summable Fredholm module over $\mathbf{A}(\mathscr{D})$. Clearly, both of the maps $p \mapsto (\mathrm{Ch}^*(\mathscr{D}), \mathrm{Ch}_*(p))$ and $p \mapsto \mathrm{ind}(\mathscr{D}_p)$ from $K_0(\mathbf{A})$ to $\mathbf{Z}$ factor through $\mathbf{A}(\mathscr{D})$ in the diagram

$$K_0(\mathbf{A}) \to K_0(\mathbf{A}(\mathscr{D})) \to \mathbf{Z}.$$

The following lemma is due to Kaplansky (see [1, Proposition 4.6.2]).

LEMMA 3.2. *If $\mathbf{A}$ is a Banach $*$-algebra in which $1 + a^*a$ is invertible for any $a \in \mathbf{A}$, any class in $K_0(\mathbf{A})$ may be represented by a self-adjoint idempotent $p \in M_r(\mathbf{A})$.*

This lemma applies in our situation: to show that $1 + a^*a$ is invertible in $\mathbf{A}(\mathscr{D})$, we use that $(1 + a^*a)^{-1}$ is a bounded operator on $\mathbf{H}$, with $\|(1 + a^*a)^{-1}\| \leqslant 1$, and

$$\begin{aligned} \|[\mathscr{D}, (1 + a^*a)^{-1}]\| &\leqslant \|(1 + a^*a)^{-1}[\mathscr{D}, a^*a](1 + a^*a)^{-1}\| \\ &\leqslant 2\|a\| \cdot \|[\mathscr{D}, a]\| < \infty. \end{aligned}$$

Thus, at the cost of replacing $\mathbf{A}$ by $\mathbf{A}(\mathscr{D})$, we may assume in proving Theorem D that the idempotent p is self-adjoint, and thus that the operator $\mathscr{D}_p$ is self-adjoint.

To finish the proof, we observe that the homotopy

$$\mathscr{D}_\tau = \mathscr{D} + \tau(2p - 1)[\mathscr{D}, p],$$

between the operators $\mathscr{D}_0 = \mathscr{D}$ and $\mathscr{D}_1 = p \cdot \mathscr{D} \cdot p + (1 - p) \cdot \mathscr{D} \cdot (1 - p)$, leads to a homotopy of Fredholm modules $(\mathbf{H}, \mathscr{D}_\tau)$ over $\mathbf{A}(\mathscr{D})$ in the sense of Corollary 2.5(1). Since $[\mathscr{D}_1, p]$ vanishes, this shows that we may choose $\mathscr{D}$ in such a way that it commutes with the idempotent p. This, we have succeeded in reducing the general case of Theorem D to the special one treated at the start of this section.

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